Complex Numbers in the UNNS Substrate: Derived Convenience or Foundational Necessity?

Abstract

UNNS (Unbounded Nested Number Sequences) models mathematics and physics via recursive operators acting on nested numeric states. Do such systems *require* complex numbers, or can complex behavior be reconstructed from integer–algebraic lattices and cyclotomic embeddings? We argue that complex numbers are not foundational for UNNS: oscillations, phases, and spectra can be realized within algebraic integer rings and their cyclotomic fields. Nevertheless, $\mathbb{C}$ offers an efficient *shorthand* for resonance and spectral geometry. We formalize embeddings, projection operators (inlaying), bounds for Gaussian and Eisenstein lattices, and show when $\mathbb{C}$ is convenient, when it is optional, and how it interfaces with UNNS thermodynamics, gauge phases, and discrete field theories.

1 UNNS Structures and Operators (Brief)

A UNNS structure is a quadruple $\mathcal{S} = (A, \mathcal{O}, \mathcal{N}, \mathcal{R})$ with seeds A, operators $\mathcal{O}$ (Collapse, Inlaying, Inletting, Normalize, Evaluate, Adopt), nesting depth $\mathcal{N}$ (recursion index), and a resonance/stability map $\mathcal{R}$. Time corresponds to recursion depth; space to lattice embedding.

2 Complex Numbers as Lattice Geometry

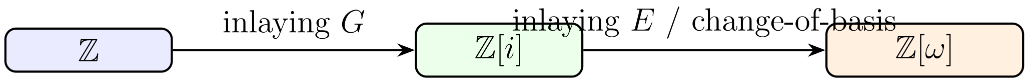
2.1 Integer lattices and algebraic lattices

Definition 1 (Algebraic lattices). *The Gaussian integers $\mathbb{Z}[i]$ form a square lattice in $\mathbb{R}^2$ and the Eisenstein integers $\mathbb{Z}[\omega]$, $\omega = e^{2\pi i/3}$ form a hexagonal lattice. More generally, rings of integers $\mathcal{O}_K$ in cyclotomic fields $K = \mathbb{Q}(\zeta_n)$ embed as full-rank lattices in $\mathbb{R}^{\phi(n)}$.*

Definition 2 (Inlaying (projection)). *Given $x \in \mathbb{R}^2$, the Gaussian inlaying $G : \mathbb{R}^2 \to \mathbb{Z}[i]$ maps x to its nearest lattice point in $\mathbb{Z}[i]$ (componentwise rounding). The Eisenstein inlaying $E : \mathbb{R}^2 \to \mathbb{Z}[\omega]$ maps x to the nearest hexagonal lattice point.*

Proposition 1 (Projection error bounds). *For Gaussian inlaying G, $\|x - G(x)\|_\infty \le \frac{1}{2}$ for all $x \in \mathbb{R}^2$. For Eisenstein inlaying E, $\|x - E(x)\|_2 \le r$, where r is the circumradius of the hexagonal Voronoi cell (a fixed numerical constant). Thus inlaying perturbs values by a uniformly bounded amount.*

2.2 Embedding chain and operator view



Remark 1. *Complex numbers appear here as coordinates on algebraic lattices. They serve as a convenient parameterization, not as an axiom of the substrate.*

3 Cyclotomic Phases and UNNS Oscillations

3.1 Roots of unity as discrete rotations

Definition 3 (Cyclotomic inlaying)**.** *For $n \geq 1$, let $\mu_n = \{\zeta_n^k : 0 \leq k < n\}$ be the n-th roots of unity. Define the cyclotomic snap $\Pi_n : \mathbb{C} \to \mu_n$ by projecting any $z \neq 0$ to the nearest ζ_n^k by angle.*

Lemma 1 (Finite-phase capture)**.** *Any finite-phase recursion (periodic phase increments) can be represented within a cyclotomic field $\mathbb{Q}(\zeta_n)$ for suitable n; its values lie in the ring of integers $\mathbb{Z}[\zeta_n]$ up to bounded projection error.*

Proof (sketch). Phase increments rational in 2π land in a cyclotomic subgroup; values are polynomials in ζ_n with integer coefficients (after clearing denominators), hence lie in $\mathbb{Z}[\zeta_n]$ up to scaling. Projection errors are uniformly bounded. □

3.2 Oscillations without invoking $\mathbb{C}$ as primitive

Proposition 2 (2D real recursion for complex eigenpairs)**.** *Let $a_{k+1} = \alpha a_k - \beta a_{k-1}$ with $\alpha, \beta \in \mathbb{Q}$ and $\alpha^2 - 4\beta < 0$. Then the complex eigenvalues $\lambda_\pm = \rho e^{\pm i\theta}$ give rise to a* real 2×2 *rotation–dilation:*

$$\begin{bmatrix} a_{k+1} \\ a_k \end{bmatrix} = \underbrace{\begin{bmatrix} \alpha & -\beta \\ 1 & 0 \end{bmatrix}}_{R(\rho,\theta)} \begin{bmatrix} a_k \\ a_{k-1} \end{bmatrix}, \quad R = \rho \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad \textit{in a suitable basis.}$$

Thus oscillations are realizable by real UNNS dynamics; $\mathbb{C}$ is a compact encoding.

4 UNNS Projection Pipelines and Spectral Stability

Let T be a linear recursion on $\mathbb{R}^d$ with spectral radius $\rho(T) > 1$. Consider a UNNS pipeline with inlaying and collapse:

$$x_{k+1} \; = \; \mathcal{T}(x_k) \; := \; \underbrace{\Pi\big(C_\varepsilon(Tx_k)\big)}_{\text{project \& repair}},$$

where Π is a fixed lattice projection (e.g. G or E), and C_ε zeros strict subthreshold values.

Lemma 2 (Uniform perturbation)**.** *There exists a norm and constant $B > 0$ such that $\|\mathcal{T}(x) - Tx\| \le B$ for all x. In particular, for $k \ge 0$, $x_k = T^k x_0 + \sum_{j=0}^{k-1} T^{k-1-j}\Delta_j$ with $\|\Delta_j\| \le B$.*

Theorem 1 (Asymptotic spectral stability)**.** *If T is diagonalizable with simple dominant eigenvalue $\lambda_\star$ and $|\lambda_\star| > \max_{i\ne\star}|\lambda_i|$, then the UNNS pipeline satisfies*

$$\lim_{k\to\infty} \frac{\|x_{k+1}\|}{\|x_k\|} = |\lambda_\star|, \qquad \frac{x_k}{\|x_k\|} \to v_\star/\|v_\star\|,$$

i.e. growth factor and projective direction of the dominant eigenpair are preserved.

Proof (outline). Standard perturbation: the additive $O(1)$ error is negligible compared to $\|T^k x_0\| \asymp |\lambda_\star|^k$; projective convergence follows from spectral dominance. □

5 Gauge Phases and Discrete Connections in UNNS

5.1 Phase inletting as $U(1)$ connection

Definition 4 (Phase inletting)**.** *Let $U_\theta = \mathrm{diag}(e^{i\theta_1}, \dots, e^{i\theta_m})$ act on a block decomposition of $x \in \mathbb{C}^m$ (or its real lift). The phase inletting step is $x \mapsto U_\theta x$ before projection Π.*

Proposition 3 (Spectral invariance under constant phase)**.** *If U_θ is constant across steps, then the spectral radius of the effective transfer is unchanged: $\rho(U_\theta T) = \rho(T)$; projective limits are rotated but stable.*

Remark 2. *Slowly varying phases (adiabatic inletting) modulate transients but preserve the dominant growth rate; fast variations imprint quasi-random residues that are absorbed by collapse/repair unless they resonate with subdominant modes.*

6 When $\mathbb{C}$ is Useful (but Optional)

6.1 Spectral geometry and concise notation

- Complex exponentials compactly represent rotations: $e^{i\theta}$.
- Eigenvalues naturally live in $\mathbb{C}$; describing spectra and resonance is simplest in $\mathbb{C}$.
- Fourier/character sums (Gauss, Jacobi, Eisenstein) enter as arithmetic constants on UNNS lattices; $\mathbb{C}$ packages their phases.

6.2 Hilbert embeddings and measurement

Definition 5 (UNNS Hilbert space)**.** *Associate to a UNNS structure $\mathcal{S}$ the Hilbert space $\mathcal{H}_\mathcal{S} = \mathrm{span}_\mathbb{C}\{|a,n\rangle : a \in A,\ n \in \mathbb{N}\}$, with $\langle a,n|a',n'\rangle = \delta_{a,a'}\delta_{n,n'}$. Operators in $\mathcal{O}$ lift to linear maps $\widehat{O}$ on $\mathcal{H}_\mathcal{S}$.*

Proposition 4. *If O is norm-preserving under $\mathcal{R}$, then $\widehat{O}$ is (approximately) unitary; collapse/repair maps are contractions. Spectra of $\widehat{O}$ encode UNNS resonance constants.*

7 Worked Examples

7.1 Fibonacci under Gaussian inlaying

Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, $v_{k+1} = G(C_\varepsilon(Av_k))$. Then $|\lambda_\star| = \varphi = (1+\sqrt{5})/2$ governs growth, and the projective direction approaches the φ-eigenline. Rounding error per step is bounded by $\frac{1}{2}$ in $\|\cdot\|_\infty$.

7.2 Complex eigenpair without $\mathbb{C}$

Let $R = \rho \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, $x_{k+1} = \Pi(C_\varepsilon(Rx_k))$. Then $\lim \|x_{k+1}\|/\|x_k\| = \rho$, and the rotation angle θ is visible in the phase of the 2D recursion even if we never mention $\mathbb{C}$.

8 Do We *Need* Complex Numbers?

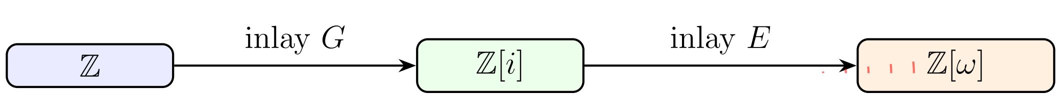
Theorem 2 (Complex as a derived envelope)**.** *Let $\mathcal{C} = \{\mathbb{Q}(\zeta_n) : n \geq 1\}$ be the filtered system of cyclotomic fields, with embeddings $\mathbb{Q}(\zeta_m) \hookrightarrow \mathbb{Q}(\zeta_{mn})$. For any finite-phase UNNS process with rational coefficients and bounded inlaying error, there exists n such that the process is representable inside $\mathbb{Q}(\zeta_n)$ (up to bounded projection). Moreover, the collection of all such representations is dense on the unit circle in the angular variable.*

Proof (sketch). Finite-phase increments land in a finite subgroup of $\mathbb{T}$, hence in μ_n for some n. Values are algebraic over $\mathbb{Q}$; after clearing denominators we obtain elements of $\mathbb{Z}[\zeta_n]$. Density follows from $\bigcup_n \mu_n$ being dense in $\mathbb{T}$ in the angular sense for rational approximations. □

Corollary 1. *For UNNS pipelines built from algebraic operators and cyclotomic inlayings, $\mathbb{C}$ is not foundational but serves as a convenient completion capturing limits and spectra succinctly.*

9 Diagrams

Embedding chain and spiral phases



cyclotomic phases on unit circle

10 Interfaces with UNNS Physics

10.1 Time-harmonic fields and DEC/FEEC

Time-harmonic fields $e^{i\omega t}$ are conveniently expressed in $\mathbb{C}$. In UNNS, ω-phases can be approximated cyclotomically; FEEC/DEC estimates carry over since inlaying errors are bounded and do not affect stability constants at leading order, while complex notation remains a concise calculus for waves.

10.2 Thermodynamics and entropy of phases

Phase dispersion under inlaying contributes to effective *entropy* in UNNS thermodynamics; collapse acts as dissipation, while constant phase inletting is entropy-neutral (spectrally inert).

11 Conclusions

- Complex numbers are *not* foundational for UNNS. Oscillations, phases, and spectra arise from algebraic integer lattices and cyclotomic inlayings.
- $\mathbb{C}$ is an *efficient shorthand*: it packages rotations, spectra, and wave phenomena succinctly, and is invaluable for analysis and exposition.
- In practice, one may implement UNNS pipelines purely over $\mathbb{Z}[i]$, $\mathbb{Z}[\omega]$, or $\mathbb{Z}[\zeta_n]$, and only lift to $\mathbb{C}$ for spectral summaries, Fourier views, or Hilbert embeddings.